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On the Simultaneous Approximation of Functions and Their Derivatives by the Szász–Mirakyan Operator*

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1. INTRODUCTION

Let f be a function defined on the interval $[0, \infty)$. The Szász–Mirakyan operator $S_n(f, x)$ is defined as follows:

$$S_n(f, x) = \sum_{k=0}^{\infty} f(k/n)\, p_k(nx), \qquad p_k(t) = e^{-t} t^k / k!. \tag{1.1}$$

Several authors (see [1–3]) studied the convergence of the operator (1.1). Recently, Fuhua Cheng [4] established an estimate of the rate of convergence for functions of bounded variation on every finite subinterval of $[0, \infty)$ and proved that if $f(t) = O(t^{\alpha t})(t \to \infty)$ for some $\alpha > 0$, then

$$\begin{aligned} &|S_n(f, x) - (1/2)(f(x+) + f(x-))| \\ &\quad \leqslant (3+x)/(nx) \sum_{k=1}^{n} V(g_x, [x - x/\sqrt{k}, x - x/\sqrt{k}]) \\ &\qquad + O(1/\sqrt{nx}) \\ &\qquad \times (|f(x+) - f(x-)| + (4x)^{4\alpha x} (e/4)^{nx}), \quad \forall x \in (0, \infty), \end{aligned} \tag{1.2}$$

where $V(g, [a, b])$ is the total variation of g on $[a, b]$ and

$$g_x(t) := \begin{cases} f(t) - f(x+), & x < t < \infty \\ 0, & t = x \\ f(t) - f(x-), & 0 \leqslant t < x. \end{cases}$$

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In a seminar lecture Meiqin Wang gave the rate of convergence of the Bernstein operator by means of a sequence of pointwise moduli of continuity improving Cheng's result [5]. The author [6] generalized Cheng's result (1.2) for the function of bounded variation of order $p \geq 1$ (BV_p) and established an estimate for $f'(x) \in BV_p$ on every finite subinterval of $[0, \infty)$.

In this paper we consider the class of functions $B_r^{(\alpha)}$ which is larger than the class of functions of generalized bounded variation

$$B_r^{(\alpha)} = \{f \mid f^{(r-1)} \in C[0, \infty),\ f_\pm^{(r)}(x) \text{ exist everywhere and are bounded on every finite subinterval of } [0, \infty) \text{ and } f_\pm^{(r)}(t) = O(t^{\alpha t})(t \to \infty) \text{ for some } \alpha > 0\} \quad (r = 0, 1, ...),$$

where $f_\pm^{(0)}(x)$ means $f(x \pm)$.

We shall prove that for $f \in B_r^{(\alpha)}$

$$|S_n(f, x) - (1/2)(f_+^{(r)}(x) + f^{(r)}(x))| \leq (73\, \Delta(x)/n) \sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k}) + 73 \sqrt{\Delta(x)/n}\, w_x(x+3) + O(e^{-cn} + |f_+^{(r)}(x) - f_-^{(r)}(x)|/(1 + \sqrt{nx})), \tag{1.3}$$

where $w_x(t) = w_x(h_r, t) = \sup\{|h_r(x+s) - h_r(x)|,\ |s| \leq t\}$ and

$$h_r(x) := \begin{cases} f_+^{(r)}(t) - f_+^{(r)}(x), & x < t < 0 \\ 0, & t = x \\ f_-^{(r)}(t) - f_-^{(r)}(x), & 0 \leq t < x. \end{cases} \tag{1.4}$$

It is clear that if g is of Λ-bounded variation [7], then

$$w_x(g, t) \leq V_\Lambda(g, [x-t, x+t]).$$

If g is continuous on $[a, b]$ and $x \in [a, b]$ then

$$w_x(g, t) \leq \omega(g, t), \tag{1.5}$$

where $\omega(g, t)$ is the usual modulus of continuity. Hence, our estimate (1.3) includes results for the function of generalized bounded variation and continuous functions. Unfortunately, our other estimate for $f \in B_{r+1}^{(\alpha)}$,

$$|S_n^{(r)}(f, x) - f^{(r)}(x)| \leq (21\, \Delta(x)/n) \times \sum_{k=1}^{n} \sqrt{\Delta(x)/k}\, w_x(\sqrt{\Delta(x)/k}) + (3/2) \times \sqrt{\Delta(x)/n}\, |f_+^{(r+1)}(x) - f_-^{(r+1)}(x)| + O(1/n),$$

does not include the case $f' \in \mathrm{Lip}\, 1$ on every finite subinterval of $[0, \infty)$. In that case we only obtain

$$S_n^{(r)}(f, x) - f^{(r)}(x) = O(\log n/n).$$

This degree is worse than the usual degree $1/n$. Therefore, the question on finding a unified estimate that includes the case $f' \in \mathrm{Lip}\, 1[0, A]$ remains open.

2. Theorems

Now we state our main results as follows.

THEOREM 1. *If $f \in B_r^{(\alpha)}$ $(r = \{0\} \cup \mathbb{N})$, then for $n \geqslant 3 + 4r^2$,*

$$\begin{aligned}|S_n^{(r)}(f, x) - (1/2)(f_+^{(r)}(x) + f_-^{(r)}(x))| &\leqslant (73\, \Delta(x)/n) \sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k}) + 73 \sqrt{\Delta(x)/n}\, w_x(x+3) \\ &\quad + O(e^{-cn} + |f_+^{(r)}(x) - f_-^{(r)}(x)|/(1 + \sqrt{nx})),\end{aligned}$$

where the sign "O" is independent of f and n but depends on x and α and $w_x(t) = w_x(h_r, t)$ is the pointwise modulus of continuity of h_r at x and h_r is defined by (1.4), $\Delta(x) = \max\{1, x\}$.

THEOREM 2. *If $f \in B_{r+1}^{(\alpha)}$ then for $x \in [0, A]$ $(A > 0)$ and $n \geqslant 4r^2$ we have*

$$\begin{aligned}|S_n^{(r)}(f, x) - f^{(r)}(x)| \leqslant{} & (21\, \Delta(x)/n) \sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k}) \sqrt{\Delta(x)/k} \\ & + (3/2)\, |f_+^{(r+1)}(x) - f_-^{(r+1)}(x)| \\ & \times \sqrt{\Delta(x)/n} + O(1/n),\end{aligned}$$

where the sign "O" is independent of x, n, and f but depends on α and A.

From Theorem 1 and Lemma 2 in Section 3, observing (1.5), we obtain the following corollaries.

COROLLARY 1. *If $f \in C[0, A]$ $(A > 0)$ and $f = O(t^{\alpha t})$ for some $\alpha > 0$, then*

$$|S_n^{(r)}(f, x) - f^{(r)}(x)| = O(\omega_{2A}(1/\sqrt{n}))$$

holds uniformly on $[0, A]$, *where*

$$\begin{aligned}\omega_A(t) &= \omega_A(f, t) \\ &= \sup\{|f(x) - f(y)|, |x - y| \leqslant t, x, y \in [0, A]\}.\end{aligned}$$

If $r=0$, this Corollary is a Theorem of Hermann [3].

COROLLARY 2. *If f is of Λ-bounded variation on every finite subinterval of $[0, \infty)$ and $f=O(t^{\alpha t})(t \to \infty)$ for some $\alpha>0$, then for $x \in (0, \infty)$*

$$\begin{aligned}
&|S_n^{(r)}(f, x)-(1/2)(f_+^{(r)}(x)+f_-^{(r)}(x))| \\
&\quad \leqslant (73\,\Delta(x)/n) \sum_{k=1}^{n} V_\Lambda(h_r, [x-\sqrt{\Delta(x)/k}, x+\sqrt{\Delta(x)/k}]) \\
&\qquad + 73 V_\Lambda(h_r, [0, 2x+3])\sqrt{\Delta(x)/n} \\
&\qquad + O(e^{-cn}+|f_+^{(r)}(x)-f_-^{(r)}(x)|/(1+\sqrt{nx})).
\end{aligned}$$

3. LEMMAS AND PRELIMINARIES

In order to prove the above theorems we need the following lemmas.

LEMMA 1. *If $f^{(r-1)} \in C[a, b]$ and $f_\pm^{(r)}(x)$ exist everywhere on $[a, b]$, then*

$$f_\pm^{(r)}(x+\theta_1 rh)h^r \leqslant \Delta_h^r f(x) \leqslant f_\pm^{(r)}(x+\theta_2 rh)h^r, \quad 0<\theta_1, \theta_2<1, \tag{3.1}$$

where the difference of order r is defined by

$$\Delta f(x)=\Delta_h^1 f(x)=f(x+h)-f(x) \text{ and } \Delta^r f(x)=\Delta_h^r f(x)=\Delta(\Delta^{r-1} f(x)).$$

Proof. On using the method of mathematical analysis it is not difficult to prove the lemma for the case $r=1$. For $r \geqslant 2$, using the mean value theorem we have

$$\begin{aligned}
\Delta^r f(x) &= \Delta^{r-1}(f(x+h)-f(x)) \\
&= [f^{(r-1)}(x+h+\theta'(r-1)h) \\
&\quad - f^{(r-1)}(x+\theta'(r-1)h)]h^{r-1} \qquad (0<\theta'<1).
\end{aligned}$$

Applying the known result for $r=1$, we obtain (3.1).

LEMMA 2. *For every $x \in [0, \infty)$ there exist constants $c>0$ and $N=N(x)$ such that*

$$\sum_{k/n \geqslant 2(x+1)} (k/n)^{\alpha k/n} p_k(nx) \leqslant e^{-cn}, \tag{3.2}$$

provided $n \geqslant N(x)$. For the interval $[0, A]$ $(A>0)$ there exist constants $c>0$ and N independent of x such that (3.2) *holds uniformly for $n \geqslant N$.*

For the proof we can follow the method of Hermann [3].
The following results are known (see [4]):

$$\sum_{k \leqslant nt} p_k(nx) \leqslant x/[n(x-t)^2], \qquad 0 \leqslant t < x \tag{3.3}$$

$$\sum_{k \geqslant nt} p(nx) \leqslant x/[n(x-t)^2], \qquad x < t \leqslant 2(x+1), \tag{3.4}$$

$$A_n(x) = \sum_{k \geqslant nx} p_k(nx) = 1/2 + O(1/(1+\sqrt{nx})), \tag{3.5}$$

$$B_n(x) = \sum_{k \leqslant nx} p_k(nx) = 1/2 + O(1/(1+\sqrt{nx})). \tag{3.6}$$

4. Proof

We only prove Theorem 2, since the proof of Theorem 1 is similar.

Proof of Theorem 2. In view of Lemma 1 and the fact that if $A \leqslant \Sigma \leqslant B$ and $\max\{|A|, |B|\} \leqslant C$, then $|\Sigma| \leqslant C$, without loss generality, it is sufficient to estimate the right side of the inequality.

Let m be a nonnegative integer such that $m/n \leqslant x < (m+1)/n$, and let $\delta = (\Delta(x)/n)^{1/2}$, $\Delta(x) = \max\{1, x\}$. Write

$$\begin{aligned} S_n^{(r)}(f, x) - f^{(r)}(x) &= \sum_{k=0}^{\infty} [n^r \Delta_{1/n}^r f(k/n) - f^{(r)}(x)]\, p_k(nx) \\ &= \sum_{k=0}^{m} [n^r \Delta^r f(k/n) - f^{(r)}(x) + f_-^{(r+1)}(x)(x-k/n)]\, p_k(nx) \\ &\quad + \sum_{k=m+1}^{\infty} [n^r \Delta^r f(k/n) - f^{(r)}(x) + f_+^{(r+1)}(x)(x-k/n)]\, p_k(nx) \\ &\quad + (1/2)(f_+^{(r+1)}(x) - f_-^{(r+1)}(x)) \sum_{k=0}^{\infty} |x-k/n|\, p_k(nx) \\ &:= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3. \end{aligned} \tag{4.1}$$

It is easy to see that

$$\left|\sum\nolimits_3\right| \leqslant (1/2)\, |f_+^{(r+1)}(x) - f_-^{(r+1)}(x)|\, (x/n)^{1/2}. \tag{4.2}$$

In order to estimate $\sum_1$ we write again

$$\sum_1 = \sum_{0 \leqslant k/n < x-\delta+(1/n)} + \sum_{x-\delta+(1/n) \leqslant k/n \leqslant m/n} = \sum_{11} + \sum_{12}. \tag{4.3}$$

Using Lemma 1 we have

$$\begin{aligned}
\sum_{11} &= \sum_{0 \leqslant k/n < x-\delta+1/n} [f^{(r)}((k+r\theta)/n) \\
&\quad - f^{(r)}(x) + f_-^{(r+1)}(x)(x-k/n)]\, p_k(nx) \\
&\leqslant \sum_{0 \leqslant k/n < x-\delta+1/n} [f_-^{(r+1)}(\xi_k) - f_-^{(r+1)}(x)](x-k/n)\, p_k(nx) \\
&\quad + \sum_{0 \leqslant k/n < x-\delta+1/n} f_-^{(r-1)}(\xi_k)\, p_k(nx)\, r\theta/n \\
&\leqslant \sum_{0 \leqslant k/n < x-\delta+1/n} w_x(x-k/n)(x-k/n)\, p_k(nx) \\
&\quad + (Mr/n) \sum_{0 \leqslant k/n < x-\delta+1/n} p_k(nx),
\end{aligned} \tag{4.4}$$

where $M = \sup_{0 \leqslant t \leqslant 2x+3} |f_\pm^{(r+1)}(t)|$, $k/n < \xi_k < x$ and $0 < \theta < 1$ but not the same at each occurrence.

Applying Abel transformation and (3.5), we obtain

$$\begin{aligned}
&\sum_{0 \leqslant k/n < x-\delta+1/n} w_x(x-k/n)(x-k/n)\, p_k(nx) \\
&\leqslant \delta w_x(\delta) \sum_{0 \leqslant k/n < x-\delta+1/n} p_k(nx) + (x/n) \\
&\quad \times \sum_{0 \leqslant k/n < x-\delta} [w_x(x-k/n)(x-k/n) \\
&\quad - w_x(x-(k+1)/n)(x-(k+1)/n)](x-k/n)^{-2} \\
&\leqslant \sqrt{\Delta(x)/n}\, w_x \sqrt{\Delta(x)/n}) \sum_{0 \leqslant k/n < x-\delta+1/n} p_k(nx) - w_x(x)/n \\
&\quad + (2x/n) \sum_{1/n \leqslant k/n < x-\delta} w_x(x-k/n)(x-k/n)^{-2}.
\end{aligned} \tag{4.5}$$

It is easy to verify that if $n \geqslant 3$,

$$\begin{aligned}
&(2x/n) \sum_{1/n \leqslant k/n < x-\delta} w_x(x-k/n)(x-k/n)^{-2} \\
&\leqslant (8x/n) \int_\delta^x w_x(t) t^{-2}\, dt \\
&= (4x/n) \int_{1/x^2}^{1/\delta^2} w_x(1/\sqrt{t})/\sqrt{t}\, dt \\
&\leqslant (4x/n) \left[\sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k})\sqrt{\Delta(x)/k} + w_x(x) \right]. \qquad (4.6)
\end{aligned}$$

A substitution of (4.5), (4.6) into (4.4) yields

$$\begin{aligned}
\left|\sum\nolimits_{11}\right| &\leqslant w_x(x)(4x+1)/n + (4x/n) \sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k})\sqrt{\Delta(x)/k} \\
&\quad + (\sqrt{\Delta(x)/n}\, w_x(\sqrt{\Delta(x)/n}) + Mr/n) \\
&\quad \times \sum_{0 \leqslant k/n < x-\delta+1/n} p_k(nx). \qquad (4.7)
\end{aligned}$$

Now turn to the estimation of $\sum_{12}$. Write

$$\sum\nolimits_{12} = \sum_{x-\delta+1/n \leqslant k/n < (m-r)/n} + \sum_{(m-r+1)/n \leqslant k/n \leqslant m/n} := \sum\nolimits_{12}' + \sum\nolimits_{12}''. \qquad (4.8)$$

Obviously

$$\left|\sum\nolimits_{12}'\right| \leqslant (w_x(\delta)\,\delta + Mr/n) \sum_{x-\delta+1/n \leqslant k/n < (m-r)/n} p_k(nx). \qquad (4.9)$$

Let

$$\begin{aligned}
J_k(x) := [&f^{(r)}((k+r\theta)/n) - f^{(r)}(x) \\
&+ f_-^{(r+1)}(x)(x-k/n)]\, p_k(nx) \qquad (m-r+1 \leqslant k \leqslant m):
\end{aligned}$$

(i) If $(k+r\theta)/n \leqslant x$,

$$\begin{aligned}
J_k(x) &\leqslant |f_-^{(r+1)}(\xi_k) - f_-^{(r+1)}(x)|\, |x-k/n| \\
&\quad \times p_k(nx) + f_-^{(r+1)}(\xi_k)\, p_k(nx)\, r\theta/n \\
&\leqslant [\sqrt{\Delta(x)/n}\, w_x(\sqrt{\Delta(x)/n}) + rM/n]\, p_k(nx).
\end{aligned}$$

(ii) If $x<(k+r\theta)/n$, then

$$J_k(x)\leqslant\{|f_+^{(r+1)}(\xi_k)-f_-^{(r+1)}(x)|\,|x-k/n|$$
$$+|f_+^{(r+1)}(x)-f_-^{(r+1)}(x)|\,|x-k/n|$$
$$+f_+^{(r+1)}(\xi_k)\,r\theta/n\}\,p_k(nx)$$
$$\leqslant\{\sqrt{\Delta(x)/n}\,w_x(\sqrt{\Delta(x)/n})+rM/n$$
$$+\sqrt{\Delta(x)/n}\,|f_+^{(r+1)}(x)-f_-^{(r+1)}(x)|\}\,p_k(nx).$$

Hence

$$\left|\sum\nolimits_{12}''\right|\leqslant(\sqrt{\Delta(x)/n}\,w_x(\sqrt{\Delta(x)/n})+rM/n)\sum_{(m-r+1/n)\leqslant k/n\leqslant m/n}p_k(nx)$$
$$+\sqrt{\Delta(x)/n}\,|f_+^{(r+1)}(x)-f_-^{(r+1)}(x)|. \tag{4.10}$$

Combining (4.8)–(4.10) we get

$$\left|\sum\nolimits_{12}\right|\leqslant(\sqrt{\Delta(x)/n}\,w_x(\sqrt{\Delta(x)/n})+rM/n)\sum_{x-\delta+1/n\leqslant k/n\leqslant m/n}p_k(nx)$$
$$+\sqrt{\Delta(x)/n}\,|f_+^{(r+1)}(x)-f_-^{(r+1)}(x)|. \tag{4.11}$$

From (4.3), (4.7), and (4.11) it follows that

$$\left|\sum\nolimits_{1}\right|\leqslant(4x/n)\sum_{k=1}^{n}w_x(\sqrt{\Delta(x)/k})\,\sqrt{\Delta(x)/k}+w_x(x)(4x+1)/n$$
$$+(\sqrt{\Delta(x)/n}\,w_x(\sqrt{\Delta(x)/n})+rM/n)\sum_{0\leqslant k\leqslant m}p_k(nx)$$
$$+\sqrt{\Delta(x)/n}\,|f_+^{(r+1)}(x)-f_-^{(r+1)}(x)|. \tag{4.12}$$

Now consider $\sum_2$. Write

$$\sum\nolimits_2=\sum_{(m+1)/n\leqslant k/n<x-\delta-(r+1)/n}+\sum_{x+\delta-(r+1)/n\leqslant k/n\leqslant 2(x+1)}+\sum_{2(x+1)<k/n}$$
$$:=\sum\nolimits_{21}+\sum\nolimits_{22}+\sum\nolimits_{23}. \tag{4.13}$$

It is evident that

$$\left|\sum\nolimits_{21}\right|\leqslant(w_x(\sqrt{\Delta(x)/n})\,\sqrt{\Delta(x)/n}+rM/n)\sum_{(m+1)/n\leqslant k/n<x+\delta-(r+1)/n}p_k(nx). \tag{4.14}$$

Similarly, if $n \geqslant 4r^2$,

$$\left|\sum\nolimits_{22}\right| \leqslant (16x/n) \sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k})\sqrt{\Delta(x)/k} + w_x(x+3)(64x+1)/n$$

$$+ (w_x(\sqrt{\Delta(x)/n})\sqrt{\Delta(x)/n} + rM/n) \sum_{x+\delta-(r+1)/n \leqslant k/n \leqslant 2(x+1)} p_k(nx). \tag{4.15}$$

Applying Lemma 2, we have

$$\sum\nolimits_{23} = O(e^{-cn}). \tag{4.16}$$

Combining (4.13)–(4.16), we get

$$\left|\sum\nolimits_{2}\right| \leqslant (16x/n) \sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k})\sqrt{\Delta(x)/k} + w_x(x+3)(64x+1)/n$$

$$+ (w_x(\sqrt{\Delta(x)/n})\sqrt{\Delta(x)/n} + rM/n)$$

$$\times \sum_{(m+1)/n \leqslant k/n \leqslant 2(x+1)} p_k(nx) + O(e^{-cn}). \tag{4.17}$$

Finally, collecting (4.1), (4.2), (4.12), and (4.17), we obtain

$$\begin{aligned}\left|\sum\nolimits_{1}\right| + \left|\sum\nolimits_{2}\right| + \left|\sum\nolimits_{3}\right| &\leqslant w_x(\sqrt{\Delta(x)/n})\sqrt{\Delta(x)/n} + rM/n \\ &\quad + (20x/n) \sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k})\sqrt{\Delta(x)/k} \\ &\quad + w_x(x+3)(68x+2)/n + (3/2)\sqrt{\Delta(x)/n} \\ &\quad \times |f_{+}^{(r+1)}(x) - f_{-}^{(r+1)}(x)| + O(e^{-cn}) \\ &\leqslant (21\,\Delta(x)/n) \sum_{k=1}^{n} w_x(\sqrt{\Delta(x)/k})\sqrt{\Delta(x)/k} \\ &\quad + (3/2)\sqrt{\Delta(x)/n}\,|f_{+}^{(r+1)}(x) - f_{-}^{(r+1)}(x)| + O(1/n).\end{aligned}$$

Observing Lemma 2, here the sign "O" is independent of x, n, and f on the interval $[0, A]$. Q.E.D.

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